Counting Techniques

Multiplication Rule 1:

In a sequence of "n" events with each event having "k" possibilities, the total number of outcomes is:

$$\mathbf{k^n = k \times k \times k \times \ldots \times k}$$

Examples:

1. **How many 6 digit student ID numbers**
   $10 \times 10 \times 10 \times 10 \times 10 \times 10 = 1\,000\,000$
2. **How many ways of throwing 5 dice (Yahtze)**
   $6 \times 6 \times 6 \times 6 \times 6 = 7\,776$
3. **How many ways of selecting 3 letters**
   $26 \times 26 \times 26 = 17\,576$

Multiplication Rule 2:

In a sequence of "n" events in which there are "k_1" possibilities for the first event, "k_2" possibilities for the second event and "k_3" for the third, etc., the total number of possible outcomes is:

$$\mathbf{k_1 \times k_2 \times k_3 \times \ldots \times k_n}$$

Examples:

4. **How many 3 digit phone exchanges before 1980**
   $8 \times 2 \times 10 = 160$
5. **How many 3 digit phone exchanges after 1980**
   $8 \times 10 \times 10 - 10$ (i.e., 011, 111, 211, ... , 911) $= 790$
6. **How many 7 character Ontario licence plates (4 letters, 3 numbers)**
   $26 \times 26 \times 26 \times 26\ 10 \times 10 \times 10 = 456\,976\,000$

Factorial Notation

Factorials:

Factorial numbers are identified with an exclamation mark (n!). They are defined:

$$\mathbf{n! = n \times n{-}1 \times n{-}2 \times \ldots \times 1}$$

0! is defined to be 1

Examples:

$1! = 1$

$2! = 2 \times 1 = 2$

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$12! = 479\ 001\ 600$

$20! \approx 2.433 \times 10^{18}$

Aside:

$$e = \sum_{i=0}^{\infty} \left(\frac{1}{i!} \right) \approx 2.71828$$

Permutations and Combinations

Permutations:

Rule 1: How many ways are there of arranging ALL "n" unique items if replacement is NOT allowed?

$$\mathbf{n! = n \times n{-}1 \times n{-}2 \times \ldots \times 1}$$

Examples:

7. **How many ways of arranging 6 items in a display.**
   $n = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

8. **How many ways of ordering three experiments.**
   $n = 3! = 6$

Rule 2: How many ways are there of selecting "r" items from "n" unique possibilities if replacement is NOT allowed?

$$\mathbf{{}_nP_r = n! / (n{-}r)! = n \times n{-}1 \times n{-}1 \times \ldots \times (n{-}r)}{-}1$$

Examples:

9. **How many ways of selecting 2 items from group of 6.**

$$nPr = \frac{6!}{4!} = \frac{6\times5\times4\times3\times2\times1}{4\times3\times2\times1} = 6\times5 = 30$$

10. **How many ways of selecting committee of four from a staff of 20 if order of selection is significant.**

$$nPr = \frac{20!}{16!} = \frac{20\times19\times18\times\ldots\times1}{16\times15\times14\times\ldots\times1} = 20\times19\times18\times17 = 116\,280$$

Permutations and Combinations cont'd

Rule 3: How many ways are there of selecting "n" items if replacement is NOT allowed but k_1, k_2, ... k_n items are identical?

$$\mathbf{n! / (k_1! \times k_2! \times \ldots \times k_n!)}$$

Examples:

11. How many words from the letters in "Ottawa".

$n = 6! / (2! \times 2!) = 180$

12. How many words from the letters in "Toronto".

$n = 7! / (2! \times 3!) = 420$

Combinations:

How many ways are there of selecting "r" items from "n" unique possibilities if replacement is NOT allowed and order is not important?

$$\mathbf{{}_nC_r = n! / [(n-r)! \times r!] = {}_nP_r / r!}$$

Examples:

13. How many ways are there of selecting 2 items from a group of 6.

$$nCr = \frac{6!}{2!\,4!} = \frac{6\times5\times4\times3\times2\times1}{(2\times1)\times(4\times3\times2\times1)} = \frac{6\times5}{2\times1} = 15$$

14. How many ways of selecting committee of four from a staff of 20 if order of selection is unimportant.

$$nCr = \frac{20!}{16!\,4!} = \frac{20\times19\times18\times17}{4\times3\times2\times1} = 4\,845$$