

Probability

Basic Concepts:

Probability experiment: process that leads to well-defined results, called **outcomes**

Outcome: result of a single trial of a probability experiment (a datum)

Sample space: all the possible outcomes of a probability experiment, i.e., the population of outcomes

Event: consists of one or more outcomes of a probability experiment, i.e., a sample of outcomes

I. Classical Probability: ratio of number of ways for an **event** to occur over total number of outcomes in the **sample space**, i.e.,

$$P(E) = \frac{n(E)}{n(S)}$$

where E is the event, S is the sample space, P(E) is the probability of the event E occurring, n(E) is the number of possible outcomes of event E, n(S) is the number of outcomes in the sample space

To obtain a probability using classical methods the event and sample space must be countable. Often the rules for determining **permutations** and **combinations** are required.

Classical Probability 1

Examples:

1. Probability of rolling “snake eyes” (1, 1) with two dice.

Sample space for rolling two dice (one white, one black):

①①	①②	①③	①④	①⑤	①⑥
②①	②②	②③	②④	②⑤	②⑥
③①	③②	③③	③④	③⑤	③⑥
④①	④②	④③	④④	④⑤	④⑥
⑤①	⑤②	⑤③	⑤④	⑤⑤	⑤⑥
⑥①	⑥②	⑥③	⑥④	⑥⑤	⑥⑥

sevens

elevens

$n(S) = 36$ possible outcomes

Notice, there are four different ways of reporting a probability (proper fraction, ratio, decimal and percentage).

$$P(1,1) = \frac{1}{36} = 1:36 = 0.0278 = 2.78\%$$

2. Probability of 7 or 11 from rolling two dice.

$$P(7 \text{ or } 11) = \frac{6+2}{36} = \frac{2}{9} = 0.222$$

3. Probability of doubles from two dice.

$$P(\text{doubles}) = \frac{6}{36} = \frac{1}{6} = 0.1667$$

- **Probability of drawing a queen from a card deck.**

$$P(\text{queen}) = \frac{4}{52} = \frac{1}{13} = 0.0769$$

Classical Probability 2

Examples:

- **Probability of drawing a spade.**

$$P(\text{spade}) = \frac{13}{52} = \frac{1}{4} = 0.250$$

- **Probability of drawing a red card.**

$$P(\text{red}) = \frac{26}{52} = \frac{1}{2} = 0.500$$

- **Probability of flipping “heads” in a coin toss.**

$$P(\text{head}) = \frac{1}{2} = 0.500$$

- **Probability of flipping “heads” after 10 coin tosses of heads in a row.**

$$P(\text{head}) = \frac{1}{2} = 0.500$$

- coin cannot “remember” its history of outcomes

- **Probability of “red” on a “double zero” roulette wheel.**

- wheel has numbers 1 to 36, half are **red** and half are **black**, plus **green** zero and double zero (n=38)

$$P(\text{red}) = \frac{18}{38} = \frac{9}{19} = 0.474$$

- **Probability of not getting a red on a roulette wheel.**

$$P(\text{not red}) = \frac{20}{38} = \frac{10}{19} = 0.526 \quad \underline{\text{or}} \quad 1 - \frac{18}{38} = 0.526$$

Rules of Probability

Rule 1: all probabilities range from 0 to 1 inclusively

$$0 \leq P(E) \leq 1$$

Rule 2: probability that an event will never occur is zero

$$P(E) = 0$$

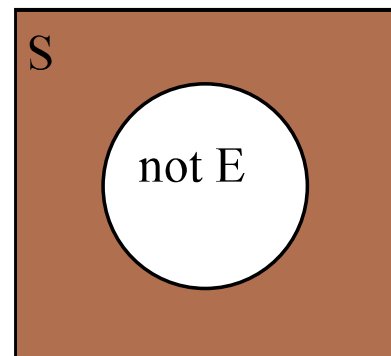
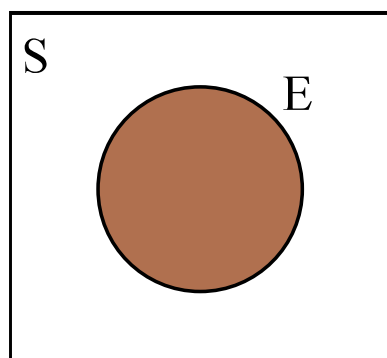
Rule 3: probability that an event will always occur is one

$$P(E) = 1$$

Rule 4: if $P(E)$ is the probability that an event will occur, the probability that the event will **not** occur is (also called the complement of an event):

$$P(\text{not } E) = 1 - P(E)$$

Venn diagrams:



Rule 5: the sum of probabilities of all outcomes in a sample space is one

$$P(S) = P(E) = 1$$

Empirical Probability

- II. Empirical Probability:** obtained empirically by sampling a population and creating a representative frequency distribution
- for a given frequency distribution, probability is the ratio of frequency of an event class to the total number of data in the frequency distribution, i.e.,

$$P(E) = \frac{\text{frequency of event class}}{\text{total data in distribution}} = \frac{f}{n}$$

Examples:

- **Probability of a girl baby.**

$$P(\text{girl}) \approx \frac{1}{2} = 0.500$$

Assume that a population has a blood type distribution of:
2% AB, 5% B, 23% A and 70% O

- **Probability of a person having type B or AB blood.**

$$P(\text{B or AB}) = 2\% + 5\% = 7.00\% = 0.0700$$

- **Probability of strongly left-handed person.**

$$P(\text{strongly left-handed}) = 0.050 = 5.00\%$$

- **Probability of “natural” blues eyes.**

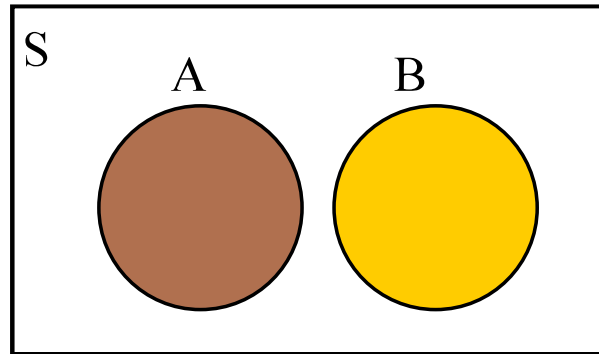
$$P(\text{blues eyes}) = 0.065 = 6.50\%$$

Addition Rules 1

Addition Rule 1: if two events are **mutually exclusive**, (i.e., no outcomes in common) then the probability of A **or** B occurring is:

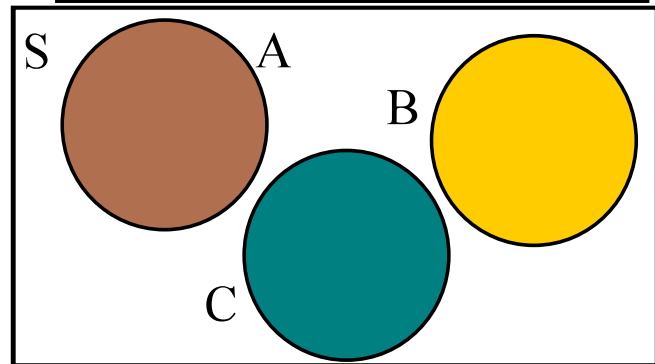
$$P(A \text{ or } B) = P(A) + P(B)$$

Venn diagrams of events that are mutually exclusive:



If **three** events are mutually exclusive:

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C)$$



Examples:

- **Probability of selecting a selecting a spade or a red card.**

$$P(\text{spade or red}) = \frac{13(S) + 26(R)}{52} = \frac{3}{4} = 0.750$$

- **Probability of drawing a face card (king or queen or jack).**

$$P(\text{face card}) = \frac{4(K) + 4(Q) + 4(J)}{52} = \frac{3}{13} = 0.231$$

- **Probability of 7, 11 or doubles with 2 dice.**

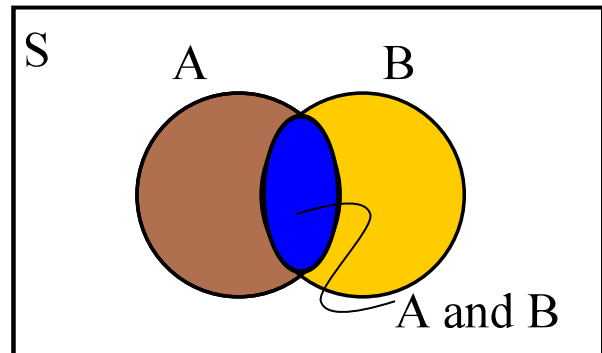
$$P(7, 11 \text{ or doubles}) = \frac{6 + 2 + 6}{36} = \frac{7}{18} = 0.389$$

Addition Rules 2

Addition Rule 2: if two events are not mutually exclusive, the probability of A or B occurring is:

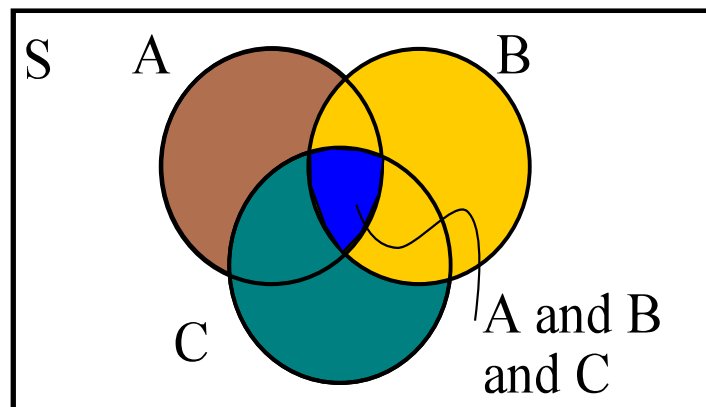
$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Venn diagrams of events that are NOT mutually exclusive:



When **three** are not mutually exclusive there is a region that is common to all three events. This area gets added and subtracted three times and therefore must be added back once. That is,

$$\begin{aligned} P(A \text{ or } B \text{ or } C) = & P(A) + P(B) + P(C) \\ & - P(A \text{ and } B) \\ & - P(B \text{ and } C) \\ & - P(A \text{ and } C) \\ & + P(A \text{ and } B \text{ and } C) \end{aligned}$$



Addition Rules 2, cont'd

Examples:

- **Probability of selecting a spade or a face card.**
 - 13 spades
 - 12 face cards (3 per suit)

$$P(\text{spade or face}) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52} = \frac{11}{26} = 0.306$$

- **Probability of selecting a female student or a third-year student from 103 students.**
 - 53 are female
 - 70 in third year
 - 45 females in third year

$$P(\text{female or year 3}) = \frac{53}{103} + \frac{70}{103} - \frac{45}{103} = \frac{78}{103} = 0.757$$

- **Probability of selecting a male or person with type O blood from 100 people.**
 - half males
 - 70% O-type blood

$$P(\text{male or type O}) = \frac{50}{100} + \frac{70}{100} - \frac{70 \times 0.5}{100} = \frac{85}{100} = 0.850$$

- **Probability of selecting a left-handed person or a Liberal from 1000 people.**
 - 32% Liberals, 5% left-handed

$$\begin{aligned} P(\text{Lib. or lefty}) &= \frac{1000 \times 0.32 + 1000 \times 0.05 - 1000 \times 0.32 \times 0.05}{1000} \\ &= \frac{354}{1000} = 0.354 \end{aligned}$$

Independence

Definition: two events are **independent** if the occurrence of one event has no effect on the occurrence of the other event. In probability experiments where there is no “memory” from one event to the next, the events are called independent.

Examples of Independent Events:

Coin tosses. Even when 10 heads are flipped in a row the next coin toss still has a 50:50 chance of being a head.

Roulette wheel spins. Each spin of the wheel is theoretically independent. Each number on the wheel has equal probability of occurring at each spin.

Rolling dice repeatedly. The dice cannot “remember” what they rolled from one toss to another.

Drawing cards with replacement. “With replacement” means after a card is drawn it is put back in the deck thus all cards are equally likely to be drawn each time.

Examples of Dependent Events:

Drawing cards without replacement. Once a card is drawn it cannot be drawn a second time. This changes the characteristics of the remaining deck of cards.

Bingo numbers. Once a ball is drawn it is not replaced.

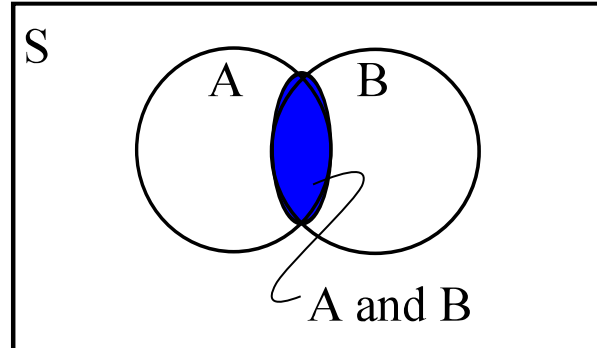
Lottery 6/49. All numbers (1-49) are equally likely to be chosen but can only be chosen once.

Multiplication Rules 1

Multiplication Rule 1: if two events are **independent** (i.e., have NO influence of each other's probability) then the probability of A **and** B occurring is:

$$P(A \text{ and } B) = P(A) \times P(B)$$

Venn diagram:



Examples:

Coin and dice tossing, lotteries, slot machines, roulette wheels etc., any game or experiment where knowledge of an outcome is not “remembered” by the next game or experiment.

- **Probability of tossing heads twice.**

$$P(2 \text{ heads}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = 0.250$$

- **Probability of rolling seven twice with two dice.**

$$P(7 \text{ twice}) = \frac{6}{36} \times \frac{6}{36} = \frac{1}{6 \times 6} = \frac{1}{36} = 0.0278$$

- **Probability of having nine daughters in a row.**

$$P(9 \text{ girls}) = \left(\frac{1}{2}\right)^9 = \frac{1}{512} = 0.001953$$

Multiplication Rules 2

Multiplication Rule 2: if two events are **dependent** then the probability of A and B occurring is:

$$P(A \text{ and } B) = P(A) \times P(B|A)$$

where **P(B|A)** means the probability of B occurring given that A occurs or occurred, also called the **conditional probability**

Examples:

Card games where the cards are not replaced or selections where replacement is not allowed. Results from one experiment affect outcome of next.

- **Probability of a drawing a two then a three.**

$$P(2 \text{ then } 3) = \frac{4}{52} \times \frac{4}{51} = \frac{16}{52 \times 51} = 0.006 \ 03$$

- **Probability of a drawing an ace then a face card.**

$$P(\text{ace then face}) = \frac{4}{52} \times \frac{12}{51} = \frac{48}{2652} = 0.018 \ 10$$

- **Probability of a drawing a pair.**

$$P(\text{pair}) = \frac{52}{52} \times \frac{3}{51} = \frac{3}{51} = 0.058 \ 8$$

- **Probability of a drawing a pair of aces.**

$$P(\text{pair aces}) = \frac{4}{52} \times \frac{3}{51} = \frac{12}{2652} = 0.004 \ 52$$