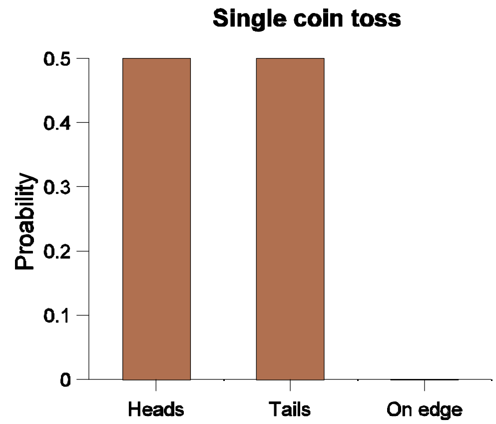
Probability Distributions

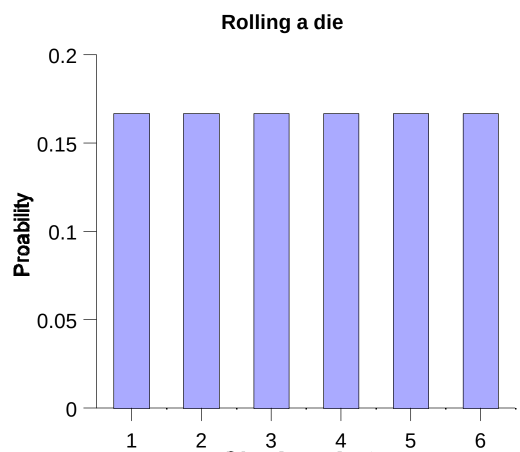
Definition: distribution of the values of a random variable and their probability of occurrence

Random variable: discrete or continuous variable whose values are determined by chance

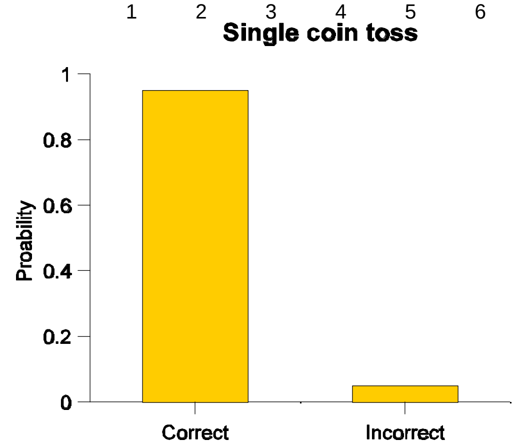
Examples:



1. Probability distribution of a coin toss (approximately 1 half)



2. Probability distribution of a "fair" die toss (each 1/6th)



3. Probability distribution of polls (correct 19 times out of 20)

Mean, Variance and Expectation

Mean: of a probability distribution (weighted average)

$$\mu = \Sigma\left[X_i \cdot P(X_i)\right]$$

where X_i is the i^{th} outcome and $P(X_i)$ is its probability

Examples:

1. Mean number of heads for tossing two coins

$$\mu = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1 \text{ (head)}$$

2. Mean number of "spots" for tossing a single die

$$\mu = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + \ldots + 6 \times \frac{1}{6} = \frac{21}{6} = 3.50 \text{ (spots!)}$$

Notice that the answer does not have to be possible.

Variance and Standard Deviation:

$$\sigma^2 = \Sigma\left[X_i^2 \cdot P(X_i)\right]$$

$$\sigma = \sqrt{\sigma^2}$$

Expectation: the expectation or expected value of a probability distribution is equal to the mean

- for predicting the cost of playing games and lotteries

$$E(X) = \mu = \Sigma\left[X_i \cdot P(X_i)\right]$$

Expectation cont'd

Examples:

1. Compute the expectation of playing a lottery where 100 tickets are sold for $1 and the winning prize is worth $100.

$$\begin{aligned} &\text{E(X)} = \$100 \times \frac{1}{100} - \$1 \\ &\text{loss / gain} = \$0.00 \end{aligned}$$

This is considered a "fair" game. If the prize was $50 the expectation would be -$0.50. Any negative value is a loser for the player; any positive value is a good game for the player.

2. Compute the profit or loss of playing a lottery where the cost of a ticket is $10, there are 1000 tickets sold and the prizes are:

   1st place wins $1000,
   2nd place wins $500 and
   five 3rd places win $100

$$\begin{aligned} &\text{E(X)} = \$1000 \times \frac{1}{1000} + \$500 \times \frac{1}{1000} + \$100 \times \frac{5}{1000} - \$10 \\ &\text{loss} = -\$8.00 \end{aligned}$$

Binomial Distribution

Definition: probability distribution in which there are only **two outcomes**, or can be reduced to only two by some rule ("an event occurs" and "the event does <u>not</u> occur")

Examples: heads and tails, true and false, success and failure, boy or girl, equal to a value and not equal, roll a "1" and not roll an "1" with a die

Rules:
- only two outcomes per trial
- fixed number of trials
- independence from trial to trial
- probability same from trial to trial

Notation:

p = probability of success
q = probability of failure
n = number of trials
X = number of successes where $0 \le X \le n$

$$\mathbf{P(X) = {}_nC_x \times p^x \times q^{n-x}}$$

Note, since $p + q = 1$ therefore $q = 1 - p$

Examples:

1. Probability of 4 sixes in 4 tosses of a die.

$$P(4\text{ sixes}) = \frac{4!}{0!4!} \times \frac{1}{6}^{4} \times \frac{5}{6}^{0} = \frac{1}{6}^{4} = 0.000\,772$$

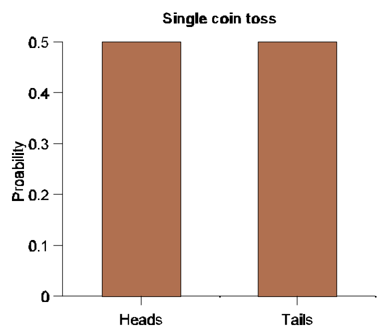
2. Probability of tossing five heads in seven tosses.

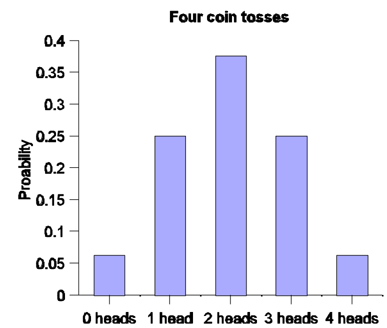
$$P(5\text{ heads}) = \frac{7!}{5!2!} \times \frac{1}{2}^{5} \times \frac{1}{2}^{2} = \frac{7 \times 6}{2 \times 1} \times \frac{1}{128} = 0.1641$$

Binomial Distributions

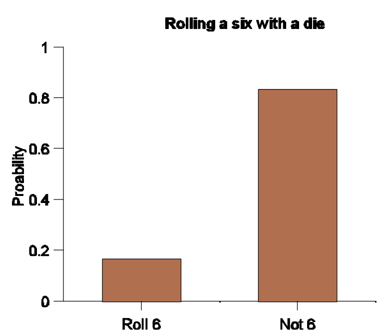
Examples:

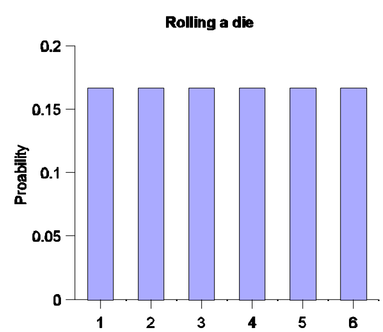
1. Tossing of a “fair” coin (1 trial and 4 trials)





2. Rolling a “six” with a fair die (rolling a die is multinomial)





3. Answering a four-choice multiple choice question correctly